$\mathcal{N} = 4$ supergravity of a T^2 - compactified sixdimensional theory

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Classical particle description

Particle physics is described by the Lagrangian $L(t, q_i, \dot{q}_i)$ and its Euler-Lagrange equation

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, i \in \{1, ..., f\},$$

where f denotes the total number of degrees of freedom. Moreover one can define an action

$$S = \int_{t_1}^{t_2} dt \; L(t, q_i, \dot{q}_i)$$

Classical field description

Define Lagrange density (called Lagrangian, too) via

$$L(t, x, \dot{x}) =: \int d^3\vec{r}\ \mathcal{L},$$

thus the action becomes

$$S = \int dt\ L = \int dt \int d^3\vec{r}\ \mathcal{L} =: \int d^4x\ \mathcal{L}$$

with $d^4x = dt\ d^3\vec{r}$.
$\rightarrow$ Go over to fourdimensional spacetime description with contravariant four-vectors $x^\mu := (t, \vec{r}), \mu \in \{0, ...3\}$

Four-vectors

In the same way define metric $\eta_{\mu\nu}, \mu, \nu \in \{0, ...3\}$, of fourdimensional Minkowski space

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\rightarrow$ Go over to covariant vectors $x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{r})$ (Einstein's sum convention: Summation over double appearing indices). Define also derivative for each component

$$\partial_\mu := \frac{\partial}{\partial x^\mu} := (\frac{\partial}{\partial t}, \vec{\nabla}), \mu \in \{0, ..., 3\}$$

Equation of motion in four spacetime dimensions

Indeed Lagrange density depends on fields, e.g.

$$\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi)$$

for scalar fields or on vectors or two-form fields, e.g.

$$\mathcal{L} = \mathcal{L}(A_\mu, \partial_\nu A_\mu)$$

Analogue to the threedimensional case one gets equation of motion for fields, e.g.

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi}$$

Example: Maxwell's inhomogeneous equations

Maxwell's inhomogeneous equations:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 4\pi\varrho(\vec{r}, t), \;\; \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

where the fields $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ are connected to a scalar potential $\Phi(\vec{r}, t)$ and a vector potential $\vec{A}(\vec{r}, t)$ via

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \;\; \vec{B} = \vec{\nabla} \times \vec{A}$$

Now define four-vectors

$$A^\mu = (\Phi, \vec{A}), \;\; j^\mu = (c\varrho, \vec{j})$$

Define field strength

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu =: \partial^{[\mu} A^{\nu]}$$

$\rightarrow$ Obtain Maxwell's equations in the form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

via the Lagrangian $\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu j^\mu$

QFT vs. GR

Two fundamental theories of the 20th century:

1. Quantum field theory: small lengths, describes particles
2. General relativity: big masses, describes universe on the whole

QFT: Origin and structure of matter

GR: Structure of space and time, history of universe

QFT vs. GR

Two fundamental theories of the 20th century:

1. Quantum field theory: small lengths, describes particles
2. General relativity: big masses, describes universe on the whole

QFT: Heisenberg's uncertainty: $\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$, $\hbar \approx 10^{-34} Js$

GR: Einstein's field equation: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

Geometry of spacetime (included in the metric $g_{\mu\nu}$ and the Riccitensor and Ricciskalar $R_{\mu\nu}, R \sim d^2 g_{\mu\nu}$) and gravity (included in the energy-momentum tensor $T_{\mu\nu}$) depend on each other.

QFT vs. GR

Two fundamental theories of the 20th century:

1. Quantum field theory: small lengths, describes particles
2. General relativity: big masses, describes universe on the whole

Both theories work well within their frontiers and are confirmed experimentally, but:

$\Rightarrow$ There is no unified theory both theories
can be deduced from (quantumgravity).
necessary in singular situations: big bang, black holes

Strings

Solution: string theory
Idea: Objects are not punctiform, but extended (strings)

typical length of a string: $l_s \approx 10^{-35} m$

Problem: Strings live in more than four spacetime dimensions, typically in $D = 10$ or $D = 26$

Compactification and dimensional reduction

Solution: Extra dimensions are rolled up. The according manifolds are for example spheres or tori.

In order to get a fourdimensional theory, one has to reduce on the according manifolds $\rightarrow$ Kaluza-Klein reduction

If one reduces on a torus T^2, seperate sixdimensional coordinates in $4+2$ coordinates:

$$\hat{x}^{\hat{\mu}} := (x^\mu, y^a),\ \hat{x}^{\hat{\mu}} \in M_6,\ x^\mu \in M_4,\ y^a \in T^2,$$

with $\mu \in \{0, ..., 3\}$ und $a \in \{4, 5\}$

Compactification and dimensional reduction

For example, consider Klein-Gordon equation in six dimensions in order to describe spin-0 particles:

$$(\hat{\Delta} - m_6^2)\hat{\Psi}(\hat{x}) = 0$$

with $\hat{\Delta} := \partial_{\hat{\mu}}\partial^{\hat{\mu}}$ as a sixdimensional Laplacian.

Set the metric to be block-diagonal

$$\rightarrow \hat{\Delta} := \Delta_4 + \Delta_2$$

with $\Delta_4 := \partial_\mu\partial^\mu$ and $\Delta_2 := \partial_a\partial^a$ as a four- and twodimensional Laplacian.

Compactification and dimensional reduction

→ Decompose $\hat{\Psi}$ according to its Fourier modes $\Psi_{n_1,n_2}(x)$
→ Inserting the decomposition of $\hat{\Psi}$ and the Laplacian one gets

$$\rightarrow 0 = (\Delta_4 - m^2_{n_1,n_2})\Psi_{n_1,n_2}(x)\ \forall n_i \in \mathbb{Z}, i \in \{1,2\}$$

where $m_{n_1,n_2} \sim \frac{n_1}{L_1} + \frac{n_2}{L_2}$.
→ Consider L_1, L_2 to be small → very massive particles (not observable)

Set

$$\Psi(x) := \Psi_{0,0}(x) := \hat{\Psi}(x^\mu, y^a)|_{y^a \equiv 0}$$

Accordingly even vectors $\hat{A}_{\hat{\mu}}$ and $\hat{B}_{\hat{\mu}\hat{\nu}}$ are said to depend only on

x^μ.→ Derivatives after internal coordinates shall vanish.

Insertion: supersymmetry

There are two kinds of particles: fermions and bosons

spin	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0
particle	graviton (?)	gravitino (?)	photon $W^{\pm}, Z^0$ gluon	lepton quark	higgs (?)

$\longrightarrow$ Fermions: build up known matter

$\longrightarrow$ Bosons: are responsible for interactions

Supersymmetric theories produce a symmetry between fermions and bosons.

Interaction particle of gravity: graviton with partner gravitino

Supergravity

Supergravity = classic, non-quantized theory which results from stringtheories if the string length goes like $l_s \longrightarrow 0$.

Theory contains a bosonic and fermionic character (here only bosonic part, fermionic part can be deduced by supersymmetry).

According to the number $x \in \mathbb{N}$ of gravitini, one speaks of $\mathcal{N} = x$ SUGRA.

$\mathcal{N} = 4, D = 6$ SUGRA

$$
\begin{aligned}
\hat{S} \quad = \quad & \int_{M_6} d^6\hat{x}\,\sqrt{-\hat{g}}\Big[\frac{1}{4}e^{-2\hat{\Phi}}\Big(-\hat{R} - 4\partial_{\hat{\mu}}\hat{\Phi}\partial^{\hat{\mu}}\hat{\Phi} + \frac{1}{3}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\varrho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\varrho}} \\
- \quad & \frac{1}{8}\partial_{\hat{\mu}}(M^{-1})_{IJ}\partial^{\hat{\mu}}M^{IJ}\Big) + \frac{1}{4}\hat{F}^{I}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}J}(M^{-1})_{IJ} + \frac{1}{2}m^I(M^{-1})_{IJ}m^J\Big] \\
+ \quad & \int_{M_6} d^6\hat{x}\,\epsilon^{\hat{\mu}\hat{\nu}\hat{\varrho}\hat{\sigma}\hat{\kappa}\hat{\lambda}}\Big(\frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{F}^{I}_{\hat{\varrho}\hat{\sigma}}\hat{F}^{J}_{\hat{\kappa}\hat{\lambda}}L_{IJ} - \frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\sigma}}\hat{F}^{J}_{\hat{\kappa}\hat{\lambda}}m^I L_{IJ} \\
+ \quad & \frac{1}{6}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\sigma}}\hat{B}_{\hat{\kappa}\hat{\lambda}}m^I L_{IJ}m^J\Big)
\end{aligned}
$$

with

$$
\begin{aligned}
\hat{\Phi} \quad &= \quad \text{dilaton} \\
\hat{R} \quad &= \quad \text{Ricci scalar} \\
\hat{B}_{\hat{\mu}\hat{\nu}} \quad &= \quad \text{spin 2-particles} \\
\hat{A}_{\hat{\mu}} \quad &= \quad \text{vector particles} \\
(M^{-1})_{IJ} \quad &= \quad \text{80 particles parameterize this matrix.}, I, J \in \{1, ..., 24\}
\end{aligned}
$$

$\mathcal{N} = 4, D = 6$ SUGRA

$$
\begin{aligned}
\hat{S} \quad &= \quad \int_{M_6} d^6\hat{x}\, \sqrt{-\hat{g}}\Big[\frac{1}{4}e^{-2\hat{\Phi}}\Big(-\hat{R} - 4\partial_{\hat{\mu}}\hat{\Phi}\partial^{\hat{\mu}}\hat{\Phi} + \frac{1}{3}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\varrho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\varrho}} \\
&- \quad \frac{1}{8}\partial_{\hat{\mu}}(M^{-1})_{IJ}\partial^{\hat{\mu}}M^{IJ}\Big) + \frac{1}{4}\hat{F}^{I}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}J}(M^{-1})_{IJ} + \frac{1}{2}m^I(M^{-1})_{IJ}m^J\Big] \\
&+ \quad \int_{M_6} d^6\hat{x}\, \epsilon^{\hat{\mu}\hat{\nu}\hat{\varrho}\hat{\sigma}\hat{\kappa}\hat{\lambda}}\Big(\frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{F}^{I}_{\hat{\varrho}\hat{\sigma}}\hat{F}^{J}_{\hat{\kappa}\hat{\lambda}}L_{IJ} - \frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\sigma}}\hat{F}^{J}_{\hat{\kappa}\hat{\lambda}}m^I L_{IJ} \\
&+ \quad \frac{1}{6}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\sigma}}\hat{B}_{\hat{\kappa}\hat{\lambda}}m^I L_{IJ}m^J\Big)
\end{aligned}
$$

Field strengths:

$$
\begin{aligned}
\hat{H}_{\hat{\mu}\hat{\nu}\hat{\varrho}} \quad &:= \quad \partial_{\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\varrho}} + \partial_{\hat{\varrho}}\hat{B}_{\hat{\mu}\hat{\nu}} + \partial_{\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\mu}}\,, \\
\hat{F}^{I}_{\hat{\mu}\hat{\nu}} \quad &:= \quad \partial_{\hat{\mu}}\hat{A}^{I}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}^{I}_{\hat{\mu}} + 2m^I\hat{B}_{\hat{\mu}\hat{\nu}}\,,
\end{aligned}
$$

and metrices

$$
\begin{aligned}
L_{IJ} \quad &= \quad \text{Metric with signature (4,20)} \\
\hat{g}_{\hat{\mu}\hat{\nu}} \quad &= \quad \text{sixdimensional metric}
\end{aligned}
$$

$\hat{g}_{\hat{\mu}\hat{\nu}}$ moves indices, e.g. $\partial_{\hat{\mu}}\hat{\Phi}\partial^{\hat{\mu}}\hat{\Phi} = \hat{g}^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\hat{\Phi}\partial_{\hat{\nu}}\hat{\Phi}$

Reduction $m \equiv 0$

Set all masses $m^I \equiv 0$ and get massless theory

$$\begin{aligned}
\hat{S}_{m\equiv 0} \;:=\; & \int_{M_6} d^6\hat{x}\,\sqrt{-\hat{g}}\Big[\frac{1}{4}e^{-2\hat{\Phi}}\Big(-\hat{R}(\hat{x}) - 4\partial_{\hat{\mu}}\hat{\Phi}\partial^{\hat{\mu}}\hat{\Phi} + \frac{1}{3}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\varrho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\varrho}} \\
- & \frac{1}{8}\partial_{\hat{\mu}}(M^{-1})_{IJ}\partial^{\hat{\mu}}M^{IJ}\Big) + \frac{1}{4}\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}J}(M^{-1})_{IJ}\Big] \\
+ & \int_{M_6} d^6\hat{x}\;\epsilon^{\hat{\mu}\hat{\nu}\hat{\varrho}\hat{\sigma}\hat{\kappa}\hat{\lambda}}\Big(\frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{F}^I_{\hat{\varrho}\hat{\sigma}}\hat{F}^J_{\hat{\kappa}\hat{\lambda}}L_{IJ}\Big)
\end{aligned}$$

with field strength

$$\hat{F}^I_{\hat{\mu}\hat{\nu}} \;:=\; \partial_{\hat{\mu}}\hat{A}^I_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}^I_{\hat{\mu}}$$

Reduction $m \equiv 0$

Decompose metric

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g^{\mu\nu} & -\sqrt{2}h^{ab}V_a^{\mu} \\ -\sqrt{2}h^{ab}V_b^{\nu} & h^{ab} + 2g^{\mu\nu}V_\mu^a V_\nu^b \end{pmatrix}$$

with

$$\begin{aligned} V_{\mu a} &= \text{Kaluza-Klein vector fields} \\ g^{\mu\nu} &= \text{fourdimensional metric} \\ h^{ab} &= \text{metric of the torus } T^2 \end{aligned}$$

One can show that $\sqrt{-\hat{g}} = \sqrt{-g}\sqrt{h}$.
Rescale the metric of the torus:

$$N_{ab} := h_{ab}e^{\varphi}$$

with $\det(N_{ab}) \equiv 1$ and new particle φ.

Reduction $m \equiv 0$

Make the following ansatzes:

$$\hat{A}^I_{\hat{\mu}} := \begin{pmatrix} \tilde{A}^I_\mu \\ A^I_a \end{pmatrix},$$

$$\hat{B}_{\hat{\mu}\hat{\nu}} := \begin{pmatrix} \tilde{B}_{\mu\nu} & \tilde{C}_{\mu b} \\ \tilde{C}_{a\nu} & B_{ab} \end{pmatrix} := \begin{pmatrix} \tilde{B}_{\mu\nu} & \tilde{C}_{\mu b} \\ \tilde{C}_{a\nu} & b\epsilon_{ab} \end{pmatrix}$$

with redefinitions:

$$F^I_\varrho := \tilde{A}^I_\varrho - \sqrt{2} V^d_\varrho A^I_d,$$

$$C_{\varrho a} := \tilde{C}_{\varrho a} + \sqrt{2} V^b_\varrho b\epsilon_{ab},$$

$$B_{\varrho\sigma} := \tilde{B}_{\varrho\sigma} + \frac{1}{\sqrt{2}} V^a_\varrho C_{\sigma a} - \frac{1}{\sqrt{2}} V^b_\sigma C_{\varrho b} - 2 V^a_\varrho V^b_\sigma b\epsilon_{ab}.$$

Reduction $m \equiv 0$

Define field strengths

$$\begin{aligned} F^I_{\mu\nu} &:= \partial_\mu F^I_\nu - \partial_\nu F^I_\mu, \\ C_{\mu\nu a} &:= \partial_\mu C_{\nu a} - \partial_\nu C_{\mu a}, \\ V^a_{\mu\nu} &:= \partial_\mu V^a_\nu - \partial_\nu V^a_\mu \end{aligned}$$

and

$$\begin{aligned} \chi &:= \Phi - \varphi, \\ \Lambda &:= \Phi + \varphi \end{aligned}$$

Separate integration according to

$$\int_{M_6} d^6\hat{x} = \int_{M_4} d^4x \int_{T^2} d^2y \text{ with } \int_{T^2} d^2y \equiv 1$$

$\longrightarrow$ Receive totally reduced massless theory

Reduction $m \equiv 0$

Reduced theory can be written as

$$S_{m=0} := S_{EH} + S_{ska} + S_1 + S_2 + S_{top}$$

with

$$S_{EH} = -\frac{1}{4}\int d^4x\sqrt{-g}R(x),$$

$$\begin{aligned} S_1 &= \int d^4x\sqrt{-g}\Big[+ \frac{1}{8}e^{\chi-\Lambda}N_{ab}V^a_{\mu\varrho}V^{\mu\varrho b} \\ &+ \frac{1}{4}e^{-\chi-\Lambda}N^{ab}(C_{\mu\varrho a} - \sqrt{2}V^c_{\mu\varrho}b\epsilon_{ac})(C^{\mu\varrho}_b - \sqrt{2}V^{\mu\varrho d}b\epsilon_{bd}) \\ &+ \frac{1}{4}e^{\chi}(F^I_{\mu\varrho} + \sqrt{2}V^c_{\mu\varrho}A^I_c)(F^{\mu\varrho J} + \sqrt{2}V^{\mu\varrho d}A^J_d)(M^{-1})_{IJ}\Big], \end{aligned}$$

$$\begin{aligned} S_{top} &= \int d^4x\epsilon^{\varrho\sigma\mu\nu}\Big[\frac{1}{2}b\epsilon_{ab}\left(\frac{1}{2}F^I_{\varrho\sigma} + \sqrt{2}\partial_\varrho(V^c_\sigma A^I_c)\right)\left(\frac{1}{2}F^J_{\mu\nu} + \sqrt{2}\partial_\mu(V^d_\nu A^J_d)\right) \\ &+ \frac{1}{2}\partial_\mu\left(B_{\varrho\sigma} - \frac{1}{\sqrt{2}}V^c_\varrho C_{\sigma c} + \frac{1}{\sqrt{2}}V^d_\sigma C_{\varrho d} + 2V^c_\varrho V^d_\sigma b\epsilon_{cd}\right)A^I_a\partial_\nu A^J_b \\ &- 2\left(C_{\varrho a} - \sqrt{2}V^b_\varrho b\epsilon_{ab}\right)\left(\frac{1}{2}F^I_{\sigma\mu} + \sqrt{2}\partial_\sigma(V^d_\mu A^I_d)\right)\partial_\nu A^J_b\Big]\epsilon^{ab}L_{IJ}, \end{aligned}$$

Reduction $m \equiv 0$

$$
\begin{aligned}
S_{ska} = \int d^4x\sqrt{-g} \quad \Big[\quad & \frac{1}{8}\partial_\mu\chi\partial^\mu\chi - \frac{1}{16}(\partial_\mu N_{ab})(\partial^\mu N^{ab}) \\
+ \quad & \frac{1}{4}N^{ab}N^{cd}e^{-2\chi}(\partial_\mu b\epsilon_{ac})(\partial^\mu b\epsilon_{bd}) \\
+ \quad & \frac{1}{2}e^{\Lambda}N^{cd}\partial_\mu A_c^I\partial^\mu A_d^J(M^{-1})_{IJ} \\
+ \quad & \frac{1}{8}\partial_\mu\Lambda\partial^\mu\Lambda - \frac{1}{32}\partial_\mu(M^{-1})_{IJ}\partial^\mu M^{IJ}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
S_2 \quad = \quad & \frac{1}{12}\int d^4x\sqrt{-g}e^{-2\Lambda}(\partial_\mu B_{\varrho\kappa} - \frac{1}{\sqrt{2}}C_{\mu a}V^a_{\varrho\kappa} - \frac{1}{\sqrt{2}}C_{\varrho\kappa a}V^a_\mu + \text{c.p.}) \\
\times \quad & (\partial^\mu B^{\varrho\kappa} - \frac{1}{\sqrt{2}}C^\mu_b V^{\varrho\kappa b} - \frac{1}{\sqrt{2}}C^{\varrho\kappa}_b V^{\mu b} + \text{c.p.})
\end{aligned}
$$

(c.p.=cyclic permutation).

Reduction $m \neq 0$

If all masses are unequal 0, regard

$$\begin{aligned}
\hat{S} \quad = & \quad \int_{M_6} d^6\hat{x}\,\sqrt{-\hat{g}}\Big[\frac{1}{4}e^{-2\hat{\Phi}}\Big(-\hat{R} - 4\partial_{\hat{\mu}}\hat{\Phi}\partial^{\hat{\mu}}\hat{\Phi} + \frac{1}{3}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\varrho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\varrho}} \\
- & \quad \frac{1}{8}\partial_{\hat{\mu}}(M^{-1})_{IJ}\partial^{\hat{\mu}}M^{IJ}\Big) + \frac{1}{4}\hat{F}^{I}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}J}(M^{-1})_{IJ} + \frac{1}{2}m^{I}(M^{-1})_{IJ}m^{J}\Big] \\
+ & \quad \int_{M_6} d^6\hat{x}\,\epsilon^{\hat{\mu}\hat{\nu}\hat{\varrho}\hat{\sigma}\hat{\kappa}\hat{\lambda}}\Big(\frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{F}^{I}_{\hat{\varrho}\hat{\sigma}}\hat{F}^{J}_{\hat{\kappa}\hat{\lambda}}L_{IJ} - \frac{1}{8}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\sigma}}\hat{F}^{J}_{\hat{\kappa}\hat{\lambda}}m^{I}L_{IJ} \\
+ & \quad \frac{1}{6}\hat{B}_{\hat{\mu}\hat{\nu}}\hat{B}_{\hat{\varrho}\hat{\sigma}}\hat{B}_{\hat{\kappa}\hat{\lambda}}m^{I}L_{IJ}m^{J}\Big)
\end{aligned}$$

$\rightarrow$ As in the massless case obtain reduced action, but this time in the form

$$S = S_{EH} + S_1 + S_2 + S_{top} + S_{ska} + S_{pot}$$

Reduction $m \neq 0$

$$S_{EH} = -\frac{1}{4}\int d^4x\sqrt{-g}R(x),$$

$$\begin{aligned} S_1 &= \frac{1}{4}\int d^4x\sqrt{-g}e^{\chi}[(F^I_{\mu\varrho}+\sqrt{2}V^a_{\mu\varrho}A^I_a)+2m^I(B_{\mu\varrho}+\sqrt{2}V^a_{[\mu}C_{\varrho]a})] \\ &\times [(F^{\mu\varrho J}+\sqrt{2}V^{\mu\varrho b}A^J_b)+2m^J(B^{\mu\varrho}+\sqrt{2}V^{[\mu b}C^{\varrho]}_b)] \\ &+ \frac{1}{8}\int d^4x\sqrt{-g}e^{\chi-\Lambda}N_{ab}V^a_{\mu\varrho}V^{\mu\varrho b} \\ &+ \frac{1}{4}\int d^4x\sqrt{-g}e^{-\chi-\Lambda}N^{ab}(C_{\mu\varrho a}-\sqrt{2}V^c_{\mu\varrho}b\epsilon_{ac})(C^{\mu\varrho}_b-\sqrt{2}V^{\mu\varrho d}b\epsilon_{bd}), \end{aligned}$$

$$\begin{aligned} S_2 &= \frac{1}{12}\int d^4x\sqrt{-g}e^{-2\Lambda}(\partial_\mu B_{\varrho\kappa}-\frac{1}{\sqrt{2}}C_{\mu a}V^a_{\varrho\kappa}-\frac{1}{\sqrt{2}}C_{\varrho\kappa a}V^a_\mu+\text{c.p.}) \\ &\times (\partial^\mu B^{\varrho\kappa}-\frac{1}{\sqrt{2}}C^\mu_b V^{\varrho\kappa b}-\frac{1}{\sqrt{2}}C^{\varrho\kappa}_b V^{\mu b}+\text{c.p.}), \end{aligned}$$

Reduction $m \neq 0$

$$
\begin{aligned}
S_{ska} &= \frac{1}{2}\int d^4x\sqrt{-g}e^{\Lambda}N^{ab}(\partial_\mu A_a^I + 2m^I C_{\mu a})(\partial^\mu A_b^J + 2m^J C_b^\mu)(M^{-1})_{IJ} \\
&+ \int d^4x\sqrt{-g}\Big[+ \frac{1}{8}\partial_\mu\Lambda\partial^\mu\Lambda - \frac{1}{16}(\partial_\mu N_{ab})(\partial^\mu N^{ab}) \\
&+ \frac{1}{4}N^{ab}N^{cd}e^{-2\chi}(\partial_\mu b\epsilon_{ac})(\partial^\mu b\epsilon_{bd}) \\
&+ \frac{1}{8}\partial_\mu\chi\partial^\mu\chi - \frac{1}{32}\partial_\mu(M^{-1})_{IJ}\partial^\mu M^{IJ}\Big],
\end{aligned}
$$

$$
\begin{aligned}
S_{pot} &= \int d^4x\sqrt{-g}\Big[+ e^{2\Lambda-\chi}N^{ab}N^{cd}b^2\epsilon_{ac}\epsilon_{bd}m^I(M^{-1})_{IJ}m^J \\
&+ \frac{1}{2}e^{\chi+2\Lambda}m^I(M^{-1})_{IJ}m^J\Big]
\end{aligned}
$$

Reduction $m \neq 0$

and

$$
\begin{aligned}
S_{top} &= \int d^4x \epsilon^{\varrho\sigma\mu\nu}\Big[\frac{1}{2}b\epsilon_{ab}\left(\frac{1}{2}F^I_{\varrho\sigma}+\sqrt{2}\partial_\varrho(V^c_\sigma A^I_c)\right)\left(\frac{1}{2}F^J_{\mu\nu}+\sqrt{2}\partial_\mu(V^d_\nu A^J_d)\right) \\
&+ \frac{1}{2}\partial_\mu\left(B_{\varrho\sigma}-\frac{1}{\sqrt{2}}V^c_\varrho C_{\sigma c}+\frac{1}{\sqrt{2}}V^d_\sigma C_{\varrho d}+2V^c_\varrho V^d_\sigma b\epsilon_{cd}\right)A^I_a\partial_\nu A^J_b \\
&- 2\left(C_{\varrho a}-\sqrt{2}V^b_\varrho b\epsilon_{ab}\right)\left(\frac{1}{2}F^I_{\sigma\mu}+\sqrt{2}\partial_\sigma(V^d_\mu A^I_d)\right)\partial_\nu A^J_b\Big]\epsilon^{ab}L_{IJ} \\
&+ \int d^4x\epsilon^{\varrho\sigma\mu\nu}\left(B_{\mu\nu}-\frac{1}{\sqrt{2}}V^c_\mu C_{\nu c}+\frac{1}{\sqrt{2}}V^d_\nu C_{\mu d}+2V^c_\mu V^d_\nu b\epsilon_{cd}\right) \\
&\times \Big[-2(C_{\varrho a}-\sqrt{2}V^c_\varrho b\epsilon_{ac})\partial_\sigma A^J_b\epsilon^{ab}m^I L_{IJ} \\
&+ 2b\left(\frac{1}{2}F^J_{\varrho\sigma}+\sqrt{2}\partial_\varrho(V^d_\sigma A^J_d)\right)m^I L_{IJ} \\
&- 2(C_{\varrho a}-\sqrt{2}V^c_\varrho b\epsilon_{ac})(C_{\sigma b}-\sqrt{2}V^d_\sigma b\epsilon_{bd})\epsilon^{ab}m^I L_{IJ}m^J \\
&+ b(B_{\varrho\sigma}-\frac{1}{\sqrt{2}}V^c_\varrho C_{\sigma c}+\frac{1}{\sqrt{2}}V^d_\sigma C_{\varrho d}+2V^c_\varrho V^d_\sigma b\epsilon_{cd})m^I L_{IJ}m^J\Big],
\end{aligned}
$$

$\mathcal{N}=4, D=4$ SUGRA

Every massless supergravity in four dimensions can be written as

$$\begin{aligned} S_{kin} \quad &:= \quad \int d^4x\sqrt{-g}\Big(\frac{1}{2}R + \frac{1}{16}\partial_\mu \mathcal{M}_{MN}\partial^\mu \mathcal{M}^{MN} - \frac{1}{4Im(\tau)^2}\partial_\mu\tau\partial^\mu\tau^* \\ &- \quad \frac{1}{4}Im(\tau)\mathcal{M}_{MN}H^{M+}_{\mu\nu}H^{\mu\nu N+}\Big) + \frac{1}{8}\int d^4x\ Re(\tau)\epsilon^{\mu\nu\varrho\sigma}\eta_{MN}H^{M+}_{\mu\nu}H^{N+}_{\varrho\sigma} \end{aligned}$$

with electric field strength $H^{M+}_{\mu\nu} := \partial_{[\mu}H^{M+}_{\nu]}$,
spin 0-particle $\tau \in \mathbb{C}, \mathcal{M}_{MN}$
and metric $\eta_{MN} \in SO(6,22)$.

→ Task: Compare this action with the result of reduction!

→ Problem: Field contents does not match!!!

Dualization of $B_{\mu\nu}$

In the general action there is no $B_{\mu\nu}$, however it is in the reduced one.
$\rightarrow$ Solution: Dualize

The general action is:

$$\mathcal{L}_2 = -\frac{f}{3!}(H_{\mu\nu\varrho} - F_{\mu\nu\varrho})(H^{\mu\nu\varrho} - F^{\mu\nu\varrho}) + \frac{1}{3!}H_{\mu\nu\varrho}J_\sigma\epsilon^{\mu\nu\varrho\sigma}$$

with

$$\begin{aligned} H_{\mu\nu\varrho}, F_{\mu\nu\varrho} &= \text{three-forms} \\ J_\sigma &= \text{vectors} \\ f &= \text{scalar function} \end{aligned}$$

Dualization of $B_{\mu\nu}$

Add Lagrange multiplier $\frac{1}{3!}H_{\mu\nu\varrho}\partial_\sigma a\epsilon^{\mu\nu\varrho\sigma}$ and use equation of motion $0=\frac{\partial\mathcal{L}_2}{\partial H_{\mu\nu\varrho}}$
Insertion of its result yields to:

$$\mathcal{L}_{2,dual}=\frac{1}{3f}g^{\mu\nu}(J_\mu+\partial_\mu a)(J_\nu+\partial_\nu a)+\frac{1}{3!}F_{\mu\nu\varrho}(J_\sigma+\partial_\sigma a)\epsilon^{\mu\nu\varrho\sigma}$$

Note:

1. The Euler-Lagrange equation for a ensures, that $H_{\mu\nu\varrho}=\partial_\mu B_{\nu\varrho}+\text{c.p.}.$
2. The scalar field a is called to be the dual field to $B_{\mu\nu}$.

Dualization of $C_{\mu a}$

Moreover in the general action there are only electric vectors. But $C_{\mu a}$,however, turns out to be a magnetic vector.
$\rightarrow$ Dualization of the vector $C_{\mu a}$

General action:

$$\mathcal{L}_1 = -\frac{f}{2} N^{ab}(C_{\mu\nu a} - J_{\mu\nu a})(C_b^{\mu\nu} - J_b^{\mu\nu}) - \frac{1}{4} C_{\mu\nu a} K^a_{\varrho\sigma} \epsilon^{\mu\nu\varrho\sigma}$$

with

$$\begin{aligned} J_{\mu\nu a}, K^a_{\varrho\sigma} &= \text{two-form fields} \\ f &= \text{scalar function} \end{aligned}$$

Dualization of $C_{\mu a}$

Add Lagrange multiplier $-\frac{1}{4}C_{\mu\nu a}D^a_{\varrho\sigma}\epsilon^{\mu\nu\varrho\sigma}$ and use Euler-Lagrange equation $0=\frac{\partial\mathcal{L}_1}{\partial C_{\mu\nu a}}$
Inserting delivers:

$$\mathcal{L}_{1,dual}=-\frac{1}{8f}N_{ab}(K^a_{\mu\varrho}+D^a_{\mu\varrho})(K^{\mu\varrho b}+D^{\mu\varrho b})-\frac{1}{4}J_{\mu\nu a}(K^a_{\varrho\sigma}+D^a_{\varrho\sigma})\epsilon^{\mu\nu\varrho\sigma},$$

where one has to use the relation between h_{ab} and N_{ab}.

Comparison

After these dualizations and after the rescaling $D^a_{\mu\nu} \to \sqrt{2} D^a_{\mu\nu}$ the topological term has the form:

$$\begin{aligned} S_{top,dual} &= \int d^4x \epsilon^{\mu\nu\varrho\sigma} \left(\frac{1}{4} b F^I_{\varrho\sigma} F^J_{\mu\nu} L_{IJ} - \frac{1}{2} V^c_{\mu\nu} b \epsilon_{ac} D^a_{\varrho\sigma} \right) \\ &= -\frac{1}{16} \int d^4x \epsilon^{\mu\nu\varrho\sigma} (-4b) \left(F^I_{\mu\nu} F^J_{\varrho\sigma} L_{IJ} - V^c_{\mu\nu} \epsilon_{bc} D^b_{\varrho\sigma} - V^d_{\mu\nu} \epsilon_{ad} D^a_{\varrho\sigma} \right) \end{aligned}$$

One can define:

$$Re(\tau) := -4b,$$

$$H^{M+}_{\mu\nu} := \begin{pmatrix} F^I_{\mu\nu} \\ D^a_{\mu\nu} \\ V^{\bar{c}}_{\mu\nu} \end{pmatrix}, \eta_{MN} := \begin{pmatrix} L_{IJ} & 0 & 0 \\ 0 & 0 & -\epsilon_{a\bar{d}} \\ 0 & \epsilon_{\bar{c}b} & 0 \end{pmatrix}, \quad M := (I, a, \bar{c})$$

Comparison

So the action can be written as

$$S_{top,dual} = -\frac{1}{16}\int d^4x Re(\tau)\eta_{MN}\epsilon^{\mu\nu\varrho\sigma}H^{M+}_{\mu\nu}H^{N+}_{\varrho\sigma}$$

$Im(\tau)$ can be determined by

$$\frac{1}{8}\frac{1}{Im(\tau)^2}\partial_\mu Re(\tau)\partial^\mu Re(\tau) = \frac{1}{4}e^{-2\chi}N^{ab}N^{cd}\partial_\mu(b\epsilon_{ac})\partial^\mu(b\epsilon_{bd})$$

with solution

$$Im(\tau) = 2e^\chi$$

$$\rightarrow \tau = \tau(b,\chi) = -4b + i2e^\chi$$

Comparison

It stays to determine $\mathcal{M}_{MN}$. For this consider complete action of vector fields and use $Im(\tau) = 2e^{\chi} \rightarrow$ obtain

$$
\begin{aligned}
\mathcal{M}_{IJ} &:= 4e^{\Lambda} N_{ab} A^K_{\bar{c}} A^L_{\bar{d}} \epsilon^{a\bar{c}} \epsilon^{b\bar{d}} L_{IK} L_{LJ} + (M^{-1})_{IJ}, \\
\mathcal{M}_{ab} &:= +2e^{\Lambda} N_{ab}, \\
\mathcal{M}_{Ib} &:= +2\sqrt{2}\ e^{\Lambda} N_{ab} A^J_{\bar{c}} \epsilon^{a\bar{c}} L_{IJ}, \\
\mathcal{M}_{a\bar{d}} &:= 2\ e^{\Lambda} N_{ab} A^I_{\bar{d}} A^J_{\bar{c}} \epsilon^{b\bar{c}} L_{IJ} - 2\ e^{\Lambda} N_{a\bar{d}} a, \\
\mathcal{M}_{I\bar{d}} &:= \sqrt{2} A^J_{\bar{d}} (M^{-1})_{IJ} + 2\sqrt{2}\ e^{\Lambda} N_{ab} A^J_{\bar{c}} A^K_{\bar{d}} A^L_{e} \epsilon^{a\bar{c}} \epsilon^{be} L_{IJ} L_{KL} \\
&\quad - 2\sqrt{2}\ e^{\Lambda} N_{a\bar{d}} A^J_{\bar{c}} a \epsilon^{a\bar{c}} L_{IJ}, \\
\mathcal{M}_{\bar{c}\bar{d}} &:= \frac{1}{2} e^{-\Lambda} N_{\bar{c}\bar{d}} + 2\ A^I_{\bar{c}} A^J_{\bar{d}} (M^{-1})_{IJ} + 2\ e^{\Lambda} N_{\bar{c}\bar{d}} a^2 \\
&\quad + 2\ e^{\Lambda} N_{ab} A^I_{\bar{c}} A^K_{\bar{d}} A^K_{e} A^L_{f} \epsilon^{ae} \epsilon^{bf} L_{IJ} L_{KL} \\
&\quad - 2\ e^{\Lambda} N_{a\bar{d}} A^I_{\bar{c}} A^J_{e} a \epsilon^{ae} L_{IJ} - 2\ e^{\Lambda} N_{\bar{c}b} A^I_{\bar{d}} A^J_{f} a \epsilon^{bf} L_{IJ}
\end{aligned}
$$

Comparison

$$\rightarrow S_{kin,1} = \frac{1}{8} \int d^4x \sqrt{-g}\, Im(\tau) \mathcal{M}_{MN} H^{M+}_{\mu\nu} H^{\mu\nu N+}$$

For the scalar fields it follows actually:

$$\begin{aligned}
-\frac{1}{32}\int d^4x\,\sqrt{-g}\partial_\mu \mathcal{M}_{MN}\partial^\mu \mathcal{M}^{MN} &= \int d^4x\sqrt{-g}\Big[-\frac{1}{16}(\partial_\mu N_{ab})(\partial^\mu N^{ab}) \\
&+ \frac{1}{2}e^\Lambda N^{cd}\partial_\mu A^I_c \partial^\mu A^J_d (M^{-1})_{IJ} \\
&+ \frac{1}{8}\partial_\mu\Lambda\partial^\mu\Lambda - \frac{1}{32}\partial_\mu (M^{-1})_{IJ}\partial^\mu M^{IJ} \\
&- \frac{2}{3}e^{2\Lambda}(A^I_a\partial_\mu A^J_b \epsilon^{ab} L_{IJ} + \partial_\mu a) \\
&\times (A^K_c\partial^\mu A^L_d\epsilon^{cd}L_{KL} + \partial^\mu a)\Big]
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{8Im(\tau)^2}\int d^4x\,\sqrt{-g}\partial_\mu\tau\partial^\mu\tau^* &= \int d^4x\sqrt{-g}\Big[\frac{1}{8}\partial_\mu\chi\partial^\mu\chi \\
&+ \frac{1}{4}N^{ab}N^{cd}e^{-2\chi}(\partial_\mu b\epsilon_{ac})(\partial^\mu b\epsilon_{bd})\Big]
\end{aligned}$$

$\mathcal{N}=4, D=4$ SUGRA

As in the massless case there is a kinetic term of the form

$$\begin{aligned} S_{kin} \quad := \quad & \int d^4x\sqrt{-g}\Big(\frac{1}{2}R + \frac{1}{16}D_\mu\mathcal{M}_{MN}D^\mu\mathcal{M}^{MN} - \frac{1}{4Im(\tau)^2}D_\mu\tau D^\mu\tau^* \\ - \quad & \frac{1}{4}Im(\tau)\mathcal{M}_{MN}H^{M+}_{\mu\nu}H^{\mu\nu N+}\Big) + \frac{1}{8}\int d^4x Re(\tau)\eta_{MN}\epsilon^{\mu\nu\varrho\sigma}H^{M+}_{\mu\nu}H^{N+}_{\varrho\sigma} \end{aligned}$$

additionally a potential

$$\begin{aligned} S_{pot} := -\frac{1}{16}\int d^4x\sqrt{-g}\Big\{ & f_{\alpha MNP}f_{\beta QRS}\tilde{M}^{\alpha\beta}\Big(\frac{1}{3}\mathcal{M}^{MQ}\mathcal{M}^{NR}\mathcal{M}^{PS} \\ & + \Big(\frac{2}{3}\eta^{MQ} - \mathcal{M}^{MQ}\Big)\eta^{NR}\eta^{PS}\Big) \\ & -\frac{4}{9}f_{\alpha MNP}f_{\beta QRS}\epsilon^{\alpha\beta}\mathcal{M}^{MNPQRS} + 3\xi^M_\alpha\xi^N_\beta\tilde{M}^{\alpha\beta}\mathcal{M}_{MN}\Big\} \end{aligned}$$

$\mathcal{N}=4, D=4$ SUGRA

and an additional topological term

$$
\begin{aligned}
S_{top} := \int \quad & - \quad \frac{1}{2}\epsilon^{\mu\nu\varrho\lambda}\Big(\xi_{+M}\eta_{NP}H_\mu^{M-}H_\nu^{N+}\partial_\varrho H_\lambda^{P+} \\
& - \quad (\hat{f}_{-MNP}+2\xi_{-N}\eta_{MP})H_\mu^{M-}H_\nu^{N+}\partial_\varrho H_\lambda^{P-} \\
& + \quad \frac{1}{16}\Theta_{+MNP}\Theta_-{}^M{}_{QR}B_{\mu\nu}^{NP}B_{\varrho\lambda}^{QR} \\
& - \quad \frac{1}{4}\hat{f}_{\alpha MNR}\hat{f}_{\beta PQ}{}^R H_\mu^{M\alpha}H_\nu^{N+}H_\varrho^{P\beta}H_\lambda^{Q-} \\
& - \quad \frac{1}{4}\Big(\Theta_{-MNP}B_{\mu\nu}^{NP}+\xi_{-M}B_{\mu\nu}^{+-}+\xi_{+M}B_{\mu\nu}^{++}\Big) \\
& \times \quad (2\partial_\varrho H_\lambda^{M-}-\hat{f}_{\alpha QR}{}^M H_\varrho^{Q\alpha}H_\lambda^{R-})\Big)
\end{aligned}
$$

$\mathcal{N}=4, D=4$ SUGRA

In comparison to the massless theory

- matrix $\tilde{M} \sim \tau$:

$$\tilde{M}_{\alpha\beta} := \frac{1}{Im(\tau)} \begin{pmatrix} |\tau|^2 & Re(\tau) \\ Re(\tau) & 1 \end{pmatrix},$$
$$\tilde{M}^{\alpha\beta} = \frac{1}{Im(\tau)} \begin{pmatrix} 1 & -Re(\tau) \\ -Re(\tau) & |\tau|^2 \end{pmatrix}, \alpha, \beta \in \{+,-\}$$

with $Im(\tau)^{-2}(\partial_\mu \tau)(\partial^\mu \tau^*) = -\frac{1}{2}(\partial_\mu \tilde{M}_{\alpha\beta})(\partial^\mu \tilde{M}^{\alpha\beta})$

- covariant derivatives:

$$D_\mu \mathcal{M}_{MN} = \partial_\mu \mathcal{M}_{MN} + 2H_\mu^{P\alpha}\Theta_{\alpha P(M}{}^{Q}\mathcal{M}_{N)Q},$$
$$D_\mu \tilde{M}_{\alpha\beta} = \partial_\mu \tilde{M}_{\alpha\beta} + H_\mu^{M\gamma}\xi_{(\alpha M}\tilde{M}_{\beta)\gamma} - H_\mu^{M\delta}\xi_{\epsilon M}\epsilon_{\delta(\alpha}\epsilon^{\epsilon\gamma}\tilde{M}_{\beta)\gamma}$$

$\mathcal{N}=4, D=4$ SUGRA

- generalised field strength:

$$\begin{aligned} H^{M+}_{\mu\nu} &:= \partial_{[\mu} H^{M+}_{\nu]} - \frac{1}{2}\hat{f}_{\alpha NP}{}^{M} H^{N\alpha}_{[\mu} H^{P+}_{\nu]} \\ &+ \frac{1}{4}\Theta_{-}{}^{M}{}_{NP} B^{NP}_{\mu\nu} + \frac{1}{4}\xi^{M}_{+} B^{++}_{\mu\nu} + \frac{1}{4}\xi^{M}_{-} B^{+-}_{\mu\nu} \end{aligned}$$

- two-form fields $B^{MN}_{\mu\nu} = B^{[MN]}_{\mu\nu}$ und $B^{\alpha\beta}_{\mu\nu} = B^{(\alpha\beta)}_{\mu\nu} = (B^{++}_{\mu\nu}, B^{+-}_{\mu\nu}, B^{--}_{\mu\nu})$ and magnetic vectors H^{M-}_{μ}
- Central quantities are the *embedding tensors* $f_{\alpha MNP} = f_{\alpha[MNP]}$, $\xi_{\alpha M}$ and their linear combinations

$$\begin{aligned} \Theta_{\alpha MNP} &:= f_{\alpha MNP} - \xi_{\alpha[N}\eta_{P]M}, \\ \hat{f}_{\alpha MNP} &:= f_{\alpha MNP} - \xi_{\alpha[M}\eta_{P]N} - \frac{3}{2}\xi_{\alpha N}\eta_{MP} \end{aligned}$$

$\mathcal{N} = 4, D = 4$ SUGRA

In the limit $f_{\alpha MNP}, \xi_{\alpha M} \rightarrow 0$ obtain massless theory

$\rightarrow$ Central task: determination of $f_{\alpha MNP}$ and $\xi_{\alpha M}$

$f_{\alpha MNP}$ and $\xi_{\alpha M}$

Consider potential

$$\begin{aligned} S_{pot} &= \int d^4x\sqrt{-g}\Big[-4e^{2\Lambda-\chi}b^2 m^I(M^{-1})_{IJ}m^J \\ &- e^{\chi+2\Lambda}m^I(M^{-1})_{IJ}m^J\Big] \end{aligned}$$

$\rightarrow$ There is only $b^2 \sim Re(\tau)^2$, but not $b \sim Re(\tau)$
$\rightarrow$ only $\tilde{M}^{--} \sim |\tau|^2 \sim Re(\tau)^2$ exists.

$$\begin{aligned} \Rightarrow f_{+MNP} &\equiv 0\ \forall M, N, P. \\ \xi_{+M} &\equiv 0\ \forall M. \end{aligned}$$

$f_{\alpha MNP}$ and $\xi_{\alpha M}$

With $\tilde{M}^{--}=\frac{|\tau|^2}{Im(\tau)}=8b^2e^{-\chi}+2e^{\chi}$ the potential takes the form

$$S_{pot}=-\frac{1}{16}\int d^4x\sqrt{-g}\quad\Big[\quad f_-^{MNP}f_-^{QRS}(8b^2e^{-\chi}+2e^{\chi})\Big(\frac{1}{3}\mathcal{M}_{MQ}\mathcal{M}_{NR}\mathcal{M}_{PS} + \Big(\frac{2}{3}\eta_{MQ}-\mathcal{M}_{MQ}\Big)\eta_{NR}\eta_{PS}\Big) + 3\xi_-^M\xi_-^N(8b^2e^{-\chi}+2e^{\chi})\mathcal{M}_{MN}\Big]$$

Difference of exponentials $\longrightarrow$ regard only first term

$$\mathcal{L}_{pot,1}:=-\frac{1}{16}f_-^{MNP}f_-^{QRS}\left(8b^2e^{-\chi}+2e^{\chi}\right)\left(\frac{1}{3}\mathcal{M}_{MQ}\mathcal{M}_{NR}\mathcal{M}_{PS}\right).$$

$f_{\alpha MNP}$ and $\xi_{\alpha M}$

$\rightarrow$ Comparison shows that $f_-^{MNP} \sim m^I$ $\rightarrow$ consider only $f_-^{Ia_1a_2}$
$\rightarrow$ Use antisymmetry and insert $\mathcal{M}$
$\rightarrow$ Comparison with potential yields to the equation

$$\begin{aligned} & - (4b^2e^{-\chi} + e^{\chi})e^{2\Lambda} m^I (M^{-1})_{IJ} m^J \\ = & - \frac{1}{16}(8b^2e^{-\chi} + 2e^{\chi}) f_-^{Ia_1a_2} f_-^{Jb_1b_2} (4e^{2\Lambda} N_{a_1b_1} N_{a_2b_2} (M^{-1})_{IJ} \\ & + 16e^{3\Lambda} N_{a_1b_1} N_{a_2b_2} N_{a_3b_3} A^K_{\bar{c}} A^L_{\bar{d}} \epsilon^{a_3\bar{c}} \epsilon^{b_3\bar{d}} L_{IK} L_{LJ} \\ & - 32e^{3\Lambda} N_{a_2b_2} N_{a_3b_1} N_{a_1b_3} A^K_{\bar{c}} A^L_{\bar{d}} \epsilon^{a_3\bar{c}} \epsilon^{b_3\bar{d}} L_{IK} L_{LJ}) \end{aligned}$$

with solution

$$\begin{aligned} f_-^{Iab} &= m^I \epsilon^{ab} \\ \Rightarrow \xi_{-M} &\equiv 0 \\ f_-^{MNP} &= \begin{cases} m^I \epsilon^{ab}, & \text{if } M = I, N = a, P = b \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$f_{\alpha MNP}$ and $\xi_{\alpha M}$

Obtain linear combinations

$$\Theta_{+MNP} = \hat{f}_{+MNP} = f_{+MNP} \equiv 0,$$

$$\Theta_{-MNP} = \hat{f}_{-MNP} = f_{-MNP} = \begin{cases} m_I \epsilon_{ab}, & \text{if } M = I, N = a, P = b \\ 0 & \text{otherwise} \end{cases}$$

Summary

1. Stringtheory lives in more than four spacetime dimensions
2. Kaluza-Klein reduction on compact manifold, e.g. twodimensional torus
3. Comparison of reduced theory with general theory
    3.1 Determine electric fields H_μ^{M+}, scalars $\mathcal{M}_{MN}$ und τ and metric η_{MN} in the massless case (also valid for the massive theory)
    3.2 Determine *embedding tensors* in the massive case
    3.3 In the limit $f_{\alpha MNP}, \xi_{\alpha M} \to 0$ obtain massless theory

Outlook

Still there is the comparison of actions in the massive case.

$\rightarrow$ Dualize once more $B_{\mu\nu}$
$\rightarrow$ In the massive case $B_{\mu\nu}$ is dual to a vector. There is a term of the form

$$\int d^4x \sqrt{-g}\; P_\mu P^\mu$$

As the field contents has to be equal to that of the massless case, set $P_\mu :\sim D_\mu a$. So

$$\int d^4x \sqrt{-g}\; P_\mu P^\mu \sim \int d^4x \sqrt{-g}\; D_\mu a D^\mu a$$

(It is not said that other summands do not appear in P_μ.)

Outlook

$\rightarrow$ In the reduced theory there will not be any two-form fields $B_{\mu\nu}$ any more.

$\rightarrow$ Eliminate B out of the general $\mathcal{N} = 4$ SUGRA

With $f_{-}^{Iab} = m^I \epsilon^{ab}$ kinetic and topological term look like:

$$
\begin{aligned}
S_{kin} &= \int d^4x\sqrt{-g}\Big[\frac{1}{2}R + \frac{1}{16}D_\mu \mathcal{M}_{MN} D^\mu \mathcal{M}^{MN} - \frac{1}{4Im(\tau)^2}\partial_\mu \tau \partial^\mu \tau^* \\
&- \frac{1}{4}Im(\tau)\mathcal{M}_{MN} H^{M+}_{\mu\nu} H^{\mu\nu N+}\Big] + \int d^4x \frac{1}{8} Re(\tau)\eta_{MN}\epsilon^{\mu\nu\varrho\sigma} H^{M+}_{\mu\nu} H^{N+}_{\varrho\sigma}
\end{aligned}
$$

Outlook

and

$$
\begin{aligned}
S_{top} = \int \quad - & \quad \frac{1}{2}\epsilon^{\mu\nu\varrho\lambda}\Big[- f_{-MNP} H_{\mu}^{M-} H_{\nu}^{N+} \partial_{\varrho} H_{\lambda}^{P-} \\
- & \quad \frac{1}{4} f_{-MNR} f_{-PQ}{}^{R} H_{\mu}^{M-} H_{\nu}^{N+} H_{\varrho}^{P-} H_{\lambda}^{Q-} \\
- & \quad \frac{1}{4} f_{-MNP} B_{\mu\nu}^{NP} (2\partial_{\varrho} H_{\lambda}^{M-} - f_{-QR}{}^{M} H_{\varrho}^{Q-} H_{\lambda}^{R-})\Big]
\end{aligned}
$$

with electric field strength

$$
\begin{aligned}
H_{\mu\nu}^{M+} & = \partial_{[\mu} H_{\nu]}^{M+} - \frac{1}{2} f_{-NP}{}^{M} H_{[\mu}^{N-} H_{\nu]}^{P+} \\
& + \frac{1}{4} f_{-}{}^{M}{}_{NP} B_{\mu\nu}^{NP} .
\end{aligned}
$$

Outlook

Use Euler-Lagrange equation for $B_{\mu\nu}^{NP}$ and obtain a new action

$$S_{kin} = \int d^4x\sqrt{-g}\Big[\frac{1}{2}R + \frac{1}{16}D_\mu \mathcal{M}_{MN} D^\mu \mathcal{M}^{MN} - \frac{1}{4Im(\tau)^2}\partial_\mu\tau\partial^\mu\tau^* - \frac{Im(\tau)}{|\tau|^2}\mathcal{M}_{MN} G_{\mu\nu}^{M-} G^{\mu\nu N-}\Big] + \int d^4x \frac{Re(\tau)}{2|\tau|^2}\eta_{MN}\epsilon^{\mu\nu\varrho\sigma} G_{\mu\nu}^{M-} G_{\varrho\sigma}^{N-}.$$

and

$$S_{top} = \int - \frac{1}{2}\epsilon^{\mu\nu\varrho\lambda}\Big(f_{-MNP} H_\mu^{M-} H_\nu^{N+}\partial_\varrho H_\lambda^{P-} + \frac{3}{4} f_{-MNR} f_{-PQ}{}^R H_\mu^{M-} H_\nu^{N+} H_\varrho^{P-} H_\lambda^{Q-}\Big)$$

with

$$G_{\mu\nu}^{M-} := \partial_{[\mu} H_{\nu]}^{M-} - \frac{1}{2} f_{-QR}{}^M H_{[\mu}^{Q-} H_{\nu]}^{R-}$$

Outlook

→ No kinetic term for electric, but only for magnetic vectors.

→ In reduced action dualize from electric vectors to magnetic ones

→ Eliminate electric vectors H_μ^{M+} from general action

→ Determine magnetic vectors H_μ^{M-} in order to finally compare both actions

Thank you :-)